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Daniel Dombek

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Uchazeč: **Ing. Daniel Dombek**
Katedra matematiky
Fakulta jaderná a fyzikálně inženýrská ČVUT v Praze
Trojanova 13, Praha 2

Školitel: **Doc. Ing. Zuzana Masáková, Ph.D.**
Katedra matematiky
Fakulta jaderná a fyzikálně inženýrská ČVUT v Praze
Trojanova 13, Praha 2

Oponenti:

prof. Christiane Frougny
Laboratoire d'Informatique Algorithmique: Fondements et Applications (LIAFA)
CNRS UMR 7089, Université Paris Diderot – Paris 7
75205 Paris Cedex 13, France

prof. Taizo Sadahiro
Department of Computer Science, Tsuda College
2-1-1 Tsuda, Kodaira City
187-8577 Tokyo, Japan

Teze byly rozeslány dne: ....

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prof. RNDr. Petr Kůrka, CSc.
předseda komise pro obhajobu disertační práce
ve studijním oboru Matematické inženýrství
Centrum pro teoretická studia, Jilská 1, 110 00 Praha 1

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1 Introduction

This thesis is devoted to the study of representations of numbers. In particular, we are interested in two major topics, firstly in positional representations of real numbers in negative real base and secondly in the possibility of representing algebraic integers in a given number field by means of algebraic units. In a particular case, we utilize a direct connection between these two representation problems.

In the first part, we study the so-called $(-\beta)$-expansions, introduced by Ito and Sadahiro [IS09] as an analogue to the β-expansions by Rényi [Rén57]. We propose a generalization of $(-\beta)$-expansions and study its properties, with the emphasis on deciding the admissibility of digit strings. Further we study the structure of the set $\mathbb{Z}_{-\beta}$ of $(-\beta)$-integers and demonstrate the exceptionality of confluent Parry numbers on the similarity of the sets of β- and $(-\beta)$-integers and their close relation to the spectrum of $(-\beta)$.

In the second part of this work we start with generalizing the so-called unit sum number problem, the problem of determining whether all algebraic integers of a given number field can be represented by sums of units. We characterize under which conditions are these representations possible. Finally, we study the so-called DUG-fields, fields with the property that all algebraic integers can be expressed as sums of distinct units.

2 Overview of the field

2.1 Rényi β-expansions

The expansions of numbers in general real base $\beta > 1$ called *β-expansions* were introduced by in [Rén57]. Defined on the unit interval $[0, 1)$ and directly extended to all reals, they led to countless interesting questions and problems, whether of combinatorial, dynamical or arithmetical nature and were extensively studied since then. Recall that, for given $z \in [0, 1)$, the β-expansion of z is the digit string $z_1z_2z_3\cdots$ obtained by the formula

$$z_i = \lfloor \beta T^{i-1}(z) \rfloor \,, \quad \text{where } T(z) = \beta z - \lfloor \beta z \rfloor \text{ is the so-called } \textit{β-transformation}\,.$$

Classification of all *admissible* digit strings, i.e. digit strings which can play the role of the β-expansion of a number from $[0, 1)$ was provided in [Par60]. Namely, it was proved that a digit string $x_1x_2x_3\cdots$ with $x_i \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ is admissible if and only if

$$x_k x_{k+1} x_{k+2} \cdots \prec_{\text{lex}} d^*_\beta(1)\,, \quad \text{for all } k \geq 1\,,$$

where $d^*_\beta(1)$ denotes the so-called *infinite Rényi expansion of unity* and $\prec_{\text{lex}}$ is the lexicographic order. Connections between algebraic properties of β and periodicity of β-expansions were studied e.g. in [Sch80].

As a natural generalization of the set $\mathbb{Z}$ of integers, the set $\mathbb{Z}_\beta$ of β-integers was defined in [BFGK98] as the set of real numbers with β-expansions of the form $x_k \cdots x_0 \bullet 0^\omega$. Even before the formal definition of β-integers, an important description of the structure of $\mathbb{Z}_\beta$ follow from earlier works [Thu89] and [Fab95].

Both $\mathbb{Z}_\beta$ and related structures, for instance the set $Fin(\beta)$ of numbers with finite β-expansions, were studied from the arithmetical point of view. For instance, in [FS92] a necessary condition for $Fin(\beta)$ being a ring was given. Moreover, the so-called *finiteness property*, given by the condition

$$Fin(\beta) = \mathbb{Z}[1/\beta]\,,$$

was also studied there.

An important set connected to $\mathbb{Z}_\beta$ is the so-called *spectrum of* β, denoted by $X^r(\beta)$ and defined in general by

$$X^r(\beta) = \Big\{ \sum_{i=0}^{N} a_i\beta^i \;\Big|\; N \in \mathbb{N}, a_i \in \{0, 1, \ldots, r\} \Big\},$$

originally studied in [EJK90]. A result from [Fro92] can be reformulated in terms of the notion of β-integers and the spectrum by saying that $X^{\lfloor\beta\rfloor} = \mathbb{Z}_\beta^+$ if and only if β is a *generalized multinacci number*, i.e. the algebraic integer with minimal polynomial of the form

$$X^d - mX^{d-1} - \ldots - mX - n\,, \qquad \text{for some } d \geq 1, m \geq n \geq 1\,.$$

In other words, for these bases β, which are often also called *confluent Parry numbers*, it holds that any linear combination of nonnegative powers of β with coefficients from $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ is a β-integer, independently on whether the corresponding sequence of coefficients is an admissible digit string or not.

Since the introduction of β-expansions, numerous generalizations have been studied, mostly defined analogously, by means of a transformation iterated on an interval with a suitably chosen digit formula. For example, [AS07] is concerned with expansions in base $\beta > 1$ defined on the interval $[-\frac{1}{2}, \frac{1}{2})$ with symmetric set of digits. More generalizations of this concept were also studied e.g. in [Gór07] and [KS12].

2.2 Ito-Sadahiro $(-\beta)$-expansions

In 2009, Ito and Sadahiro [IS09] introduced the notion of $(-\beta)$-expansions, a straightforward negative base analogue to Rényi β-expansions. Their choice of the transformation

$$T(z) = -\beta z - \lfloor -\beta z - \ell \rfloor\,, \qquad \text{for } z \in [\ell, \ell+1) \text{ where } \ell = \tfrac{-\beta}{\beta+1}\,,$$

leads to expansions of numbers from $[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ by infinite strings of digits from the alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor\}$. They also proved an analogue to the admissibility condition in base $\beta > 1$, that a digit string over $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ is admissible if and only if

$$d\big(\tfrac{-\beta}{\beta+1}\big) \preceq_{\text{alt}} x_k x_{k+1} x_{k+2} \cdots \prec_{\text{alt}} d^*\big(\tfrac{1}{\beta+1}\big)\,, \quad \text{for all } k \geq 1\,.$$

Here, $\prec_{\text{alt}}$ denotes the alternate order on infinite strings, $d\big(\frac{-\beta}{\beta+1}\big)$ is the $(-\beta)$-expansion of the left endpoint of the given unit interval and $d^*\big(\frac{1}{\beta+1}\big)$ is defined as the limit of $(-\beta)$-expansions of numbers approaching the right endpoint from the left, in fact a direct analogue to the infinite Rényi expansion of unity. The relation between these two reference strings $d\big(\frac{-\beta}{\beta+1}\big)$ and $d^*\big(\frac{1}{\beta+1}\big)$ is also given in [IS09]. Note that in contrast to the analogous result on β-expansions, it is much more complicated to decide whether a given digit string can play the role of $d\big(\frac{-\beta}{\beta+1}\big)$ for some $\beta > 1$ or not. An equivalent criterion for this was given by Steiner in [Ste13]. Numerous interesting results on $(-\beta)$-expansions also from the dynamical and ergodic points of view, were given in [IS09] as well as in many other works.

In analogy with Parry numbers, *Yrrap numbers* were defined as those $\beta > 1$ with reference string $d\big(\frac{-\beta}{\beta+1}\big)$ periodic. Connections between classes of Yrrap, Pisot and Salem numbers, as well as arithmetics in base $(-\beta)$ were studied in [FL11]

In [Kal14], the relation between Rényi β- and Ito-Sadahiro $(-\beta)$-transformation was studied. In particular, it was proved that these two are measurably isomorphic if and only if β belongs to the class of confluent Parry numbers.

Following the chain of analogies and comparisons with β-expansions when studying $(-\beta)$-expansions, the attention naturally turned to the set $\mathbb{Z}_{-\beta}$ of $(-\beta)$-integers, which represent one of the major topics of this thesis. $\mathbb{Z}_{-\beta}$ can be defined either as the set of real numbers with $(-\beta)$-expansions having zero fractional part, or by means of the $(-\beta)$-transformation $T_{-\beta}$ and its preimages of zero as

$$\mathbb{Z}_{-\beta} = \bigcup_{i\geq 0} (-\beta)^i T_{-\beta}^{-i}(0)\,.$$

The latter was chosen e.g. in [Ste12] where it is proved that if β is an Yrrap number, $\mathbb{Z}_{-\beta}$ can be encoded by an infinite word, obtainable as a fixed point of an antimorphism. Note that this antimorphism is derived solely from the dynamical properties of $(-\beta)$-transformation (using the notion of the so-called first return map) and does not work with actual $(-\beta)$-expansions of elements of $\mathbb{Z}_{-\beta}$.

Primarily from the combinatorial and arithmetical point of view, the sets $\mathbb{Z}_{-\beta}$ of $(-\beta)$-integers and $Fin(-\beta)$ of numbers with finite $(-\beta)$-expansions were studied for instance in [MPV11], where the criterion for the finiteness property was given in case that β is a quadratic Pisot number. Quadratic Pisot numbers were also studied in [MV14], where the similarity of $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ was discussed, also in connection to the spectra $X^{\lfloor\beta\rfloor}(-\beta)$.

2.3 Representations of algebraic integers by sums of units

Second representation problem considered in this thesis, the representation of algebraic integers in a given number field by its units, can be viewed as a special case of a very rich topic, in particular of *additive unit structure of rings*. Given a ring R, its *unit sum number* $u(R)$ is defined as the minimal integer k such that any element of R is a sum of at most k units of R, if such an integer exists. If it does not exist, then we put $u(R) = \omega$ if every element is a sum of units, and $u(R) = \infty$ if not. The convention $k < \omega < \infty$ is used here, for all integers k. The task of determining the unit sum number, given a ring R, is called the *unit sum number problem*. For a more detailed overview, we refer the reader to the survey paper [BFT11].

The fundamental result about the additive structure of the rings of integers in number fields is that there is no integer k such that every element of R is a sum of at most k units, i.e. $u(R) \geq \omega$. After several partial results, the proof of this statement was eventually given by Jarden and Narkiewicz in [JN07].

The so-called *qualitative problem* immediately follows from the result of Jarden and Narkiewicz. Does there exist a criterion to determine whether the unit sum number $u(O_K)$ is equal to ω or to ∞? Several criteria of this type do exist, but only for number fields of low degree. In particular, the quadratic case was solved in [AV05] while pure cubic fields were dealt with in [TZ07]. Similar results exist also for complex purely quartic fields.

When studying the additive unit structure of rings, one often finds that it is particularly useful to be able to bound the possible lengths of nontrivial arithmetic progressions in the group of units or in similar structures. This was achieved for units in [New90] and consequently generalized in [BHP10] to algebraic integers of given positive norm m.

2.4 Distinct unit generated fields

One can ask a similar question to the one connected to the unit sum number. Instead of studying whether all algebraic integers in a given field are expressible as sums of units, one may require these sums to consist of distinct units. Already in 1960's, Jacobson in [Jac64] observed that the two number fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ share the property that every algebraic integer is the sum of distinct units. Moreover, he conjectured that these two quadratic number fields are the only quadratic number fields with this property. Fields with this property are called distinct unit generated fields or *DUG-fields* for short.

This problem was solved in [Śli74] for quadratic number fields and showed that even no pure cubic number field is DUG. These results have been extended in [Bel76] to the case of cubic and quartic fields. In the latter, the complete solution to the case of imaginary cubic number fields was given. The problem of characterizing all number fields in which every algebraic integer is a sum of distinct units is, similarly to the unit sum number problem, still unsolved.

The following definition for measuring how far is a number field away from being a DUG-field was introduced in [TZ11]. Given an algebraic number field K, its *unit sum height* $\omega(K) = \omega(O_K)$ is the minimal integer k such that any element of O_K can be expressed as a linear combination of distinct units with coefficients coming from $\{0, 1, \dots, k\}$, if such an integer exists. If it does not exist, then we put $\omega(K) = \omega$ if every element is a sum of units, and $\omega(K) = \infty$ if not. Clearly, DUG-fields are then characterized by property $\omega(K) = 1$.

Moreover, the authors of [TZ11] developed a new approach for determining the unit sum height of fields with unit rank 1. They were able to determine $\omega(K)$ for all quadratic and pure cubic fields K. Nevertheless, their method was not suited for totally complex fields, which ruled out the last class of fields with unit rank 1, the totally complex quartic fields. These fields were studied separately in [HZ14] by different means and a partial result was given there. In particular, a list of fields was provided, containing all DUG-fields of this class, and it was conjectured that all fields in this list of candidates are indeed DUG.

Assume that the fundamental unit ϵ of field K with unit rank 1 also generates the integral basis of the ring of integers in, i.e. $O_K = \mathbb{Z}[\epsilon]$. Then there exists a connection between the DUG property of K and the finiteness property of ϵ, mentioned above. Both these properties can be reformulated in terms of rewriting representations in base ϵ of one given form into another one. Similar rewriting approach is used for instance in [FPS11] in the realization of parallel addition in a complex base β. There is an important consequence in a positional system allowing this parallel addition, namely that there exists a finite alphabet $\mathcal{A}$ such that the set of numbers $\sum_{i=\ell}^{k} a_i \beta^i$ with $a_i \in \mathcal{A}$ is a ring.

3 Results on the generalized $(-\beta)$-expansions

In this thesis, we propose the following generalization of $(-\beta)$-expansions, introducing a parameter $\ell \in \mathbb{R}$. Note that unlike the conventions in the positive base numeration, we distinguish expansions of numbers from the unit interval $[\ell, \ell + 1)$ from expansions of all real numbers by using a notion of $T_{-\beta,\ell}$-expansions.

Definition 1. *Let $\beta > 1$, $\ell \in \mathbb{R}$. For any $z \in [\ell, \ell + 1)$, define $d_{-\beta,\ell}(z) = z_1 z_2 z_3 \cdots$ by*

$$z_i := \lfloor -\beta T_{-\beta,\ell}^{i-1}(z) - \ell \rfloor , \quad T_{-\beta,\ell}(z) := -\beta z - \lfloor -\beta z - \ell \rfloor . \tag{1}$$

We call $T_{-\beta,\ell} : [\ell, \ell+1) \to [\ell, \ell+1)$ a ***$(-\beta, \ell)$-transformation*** *and $d_{-\beta,\ell}(z)$ a* ***$T_{-\beta,\ell}$-expansion*** *of $z \in [\ell, \ell+1)$. If the values of β, ℓ are clear from context, one may use the shorter notation $T := T_{-\beta,\ell}$, $d(x) := d_{-\beta,\ell}(x)$.*

It follows for any $d_{-\beta,\ell}(z) = z_1 z_2 z_3 \cdots$ of $z \in [\ell, \ell+1)$ that

$$z = \frac{z_1}{-\beta} + \frac{z_2}{(-\beta)^2} + \frac{z_3}{(-\beta)^3} + \dots ,$$

where the alphabet $\mathcal{A}_{-\beta,\ell}$ of $T_{-\beta,\ell}$-expansions depends on the choice of ℓ and can be calculated directly as

$$\mathcal{A}_{-\beta,\ell} := \{\lfloor -\ell(\beta+1) - \beta \rfloor, \dots, \lfloor -\ell(\beta+1) \rfloor\} . \tag{2}$$

One may impose several restrictions on the choice of parameter $\ell \in \mathbb{R}$ in order to obtain various useful properties of $T_{-\beta,\ell}$-expansions. In particular, throughout the work we assume that

$$\ell \in (-1, 0] .$$

This choice ensures that zero is a valid digit and that $d(0) = 000 \cdots = 0^\omega$. Moreover, it allows to extend the definition of expansions from $[\ell, \ell+1)$ to all real numbers in the sense analogous to the definition of β-expansions. Further requirements are summarized in the following:

- ◇ The alphabet $\mathcal{A}_{-\beta,\ell}$ of $T_{-\beta,\ell}$-expansions is equal to $\{0, 1, \dots, \lfloor \beta \rfloor\}$ if and only if

$$-\frac{\lfloor \beta \rfloor + 1}{\beta + 1} < \ell \leq -\frac{\beta}{\beta+1} .$$

- ◇ The "shift invariance" property, $d(x) = x_1 x_2 x_3 \cdots \Rightarrow d\big(\frac{x}{(-\beta)^k}\big) = 0^k x_1 x_2 x_3 \cdots$, holds if and only if

$$-\frac{\beta}{\beta+1} < \ell \leq -\frac{1}{\beta+1} .$$

- ◇ The set $\mathrm{Fin}(-\beta, \ell) = \{x \in [\ell, \ell+1) \mid T^n(x) = 0 \text{ for some } n \in \mathbb{N}\}$ of numbers with finite expansions is different from $\{0\}$ if and only if

$$-\frac{1}{\beta} \text{ or } \frac{1}{\beta} \in [\ell, \ell+1) .$$

3.1 Admissibility and reference strings

We prove an analogous result to the admissibility criteria given by Parry [Par60] for β-expansions and by Ito and Sadahiro [IS09] for $(-\beta)$-expansions. The so-called strong admissibility is introduced because it greatly simplifies the proper definition of unique expansions of real numbers outside $[\ell, \ell+1)$, as is demonstrated later in the thesis.

Definition 2. *Let $d_1 d_2 d_3 \cdots \in \mathcal{A}_{-\beta,\ell}^{\mathbb{N}}$, it is said to be* ***$(-\beta, \ell)$-admissible*** *or just* ***admissible*** *if no confusion is possible, if there exists $x \in [\ell, \ell+1)$ such that $d_{-\beta,\ell}(x) = d_1 d_2 d_3 \cdots$.*

The digit string $d_1 d_2 d_3 \cdots \in \mathcal{A}_{-\beta,\ell}^{\mathbb{N}}$ is said to be ***strongly $(-\beta, \ell)$-admissible*** *or just* ***strongly admissible****, if the digit string $0 d_1 d_2 d_3 \cdots$ is admissible.*

Theorem 3. *Denote* $d(\ell) = l_1 l_2 l_3 \cdots$ *and* $d^*(\ell+1) = r_1 r_2 r_3 \cdots$. *An infinite string* $x_1 x_2 x_3 \cdots \in \mathcal{A}_{-\beta,\ell}$ *is* $(-\beta,\ell)$*-admissible, if and only if*

$$l_1 l_2 l_3 \cdots \preceq_{\text{alt}} x_i x_{i+1} x_{i+2} \cdots \prec_{\text{alt}} r_1 r_2 r_3 \cdots , \qquad \text{for all } i \geq 1 , \tag{3}$$

where $\preceq_{\text{alt}}$ *denotes the alternate order on strings defined by*

$$x \preceq_{\text{alt}} y \ \Leftrightarrow \ x = y \text{ or } (-1)^k (x_k - y_k) < 0 \text{ for } k = \min\{i \geq 1 \mid x_i \neq y_i\} .$$

Although Theorem 3 covers a more general class of numeration systems than its analogues, it does not provide us with an explicit description of the digit string $d^*(\ell+1) = r_1 r_2 r_3 \cdots$ which plays a key role in the admissibility criterion. We call both the important digit strings

$$d(\ell) = d_{-\beta,\ell}(\ell) = l_1 l_2 l_3 \cdots , \quad d^*(\ell+1) = d^*_{-\beta,\ell}(\ell+1) = r_1 r_2 r_3 \cdots$$

from the admissibility condition the **reference strings**. The following theorem states the relation between $d(\ell)$ and $d^*(\ell) = \lim_{\epsilon \to 0^+} d(\ell + \epsilon)$, which can be of great use when determining $d^*(\ell+1)$.

Theorem 4. *Let* $\beta > 1$. *If* $T^q(\ell) \neq \ell$ *for all* $q \in \mathbb{N}$, *or the equality* $T^q(\ell) = \ell$ *occurs only for even* $q \in \mathbb{N}$ *(i.e. if* $d(\ell)$ *is not purely periodic with an odd period), then*

$$d^*(\ell) = d(\ell) .$$

If on the other hand $T^q(\ell) = \ell$ *for some* $q \in \mathbb{N}$, q *odd, i.e.,* $d(\ell) = (l_1 l_2 \cdots l_{q-1} l_q)^\omega$, *then*

$$d^*(\ell) = l_1 l_2 \cdots l_{q-1}(l_q - 1) d^*(\ell+1) .$$

An interesting problem is to determine what sequences may play the role of the left and right reference strings in the admissibility condition, which is far more difficult than the analogous question in the case of β-expansions. In this thesis, we explain the phenomenon on two examples which, in fact, represent a counterexample to Theorem 25 of Góra [Gór07] who approaches this problem in a more general setting. Note that both necessary and sufficient condition for a given word from $\mathcal{A}_{-\beta,\ell}$ to play a role of a reference string corresponding to some $\beta > 1$ and $\ell \in (-1, 0]$ was eventually completely solved by Steiner in [Ste13].

3.2 Periodicity

When studying periodic expansions of numbers in $[\ell, \ell+1) \cap \mathbb{Q}(\beta)$ in base $-\beta$, we give analogues to the results of Schmidt [Sch80] and Frougny with Lai [FL11] in the context of generalized $(-\beta)$-expansions, valid for any $\ell \in (-1, 0]$.

Theorem 5. *Let* $\beta > 1$ *and* $l \in (-1, 0]$. *If* β *is a Pisot number, then* $d(x) = d_{-\beta,\ell}(x)$ *is periodic (eventually or purely) for any* $x \in [\ell, \ell+1) \cap \mathbb{Q}(\beta)$.

Theorem 6. *Let* $\beta > 1$ *and* $\ell \in (-1, 0]$. *If any* $x \in [\ell, \ell+1) \cap \mathbb{Q}$ *has eventually periodic* $d_{-\beta,\ell}(x)$, *then* β *is either a Pisot or a Salem number.*

3.3 $(-\beta,\ell)$-expansions of real numbers

We introduce the notion of (β,ℓ)-expansions, the extension of $d_{-\beta,\ell}(x)$ to all reals, and an analogue to the notion of β-expansions. We show that in contrast with β-expansions, this extension from $[\ell,\ell+1)$ (with $\ell\in(-1,0]$) is not always given uniquely, which is illustrated on several counterexamples.

Definition 7. *Let $\beta>1$, $\ell\in(-1,0]$. Any expression of the form $y_l\cdots y_0\bullet y_{-1}\cdots$ satisfying $x=\sum_{i\le l} y_i(-\beta)^i$, $y_l\neq 0$ and $y_i\in\mathcal{A}_{-\beta,\ell}$ for all $i\le l$ is said to be a* ***$(-\beta,\ell)$-representation*** *of x.*

A $(-\beta,\ell)$-representation $y_l\cdots y_0\bullet y_{-1}\cdots$ is said to be ***admissible****, if the digit string $y_l y_{l-1}\cdots$ is admissible.*

In the positive base case, the β-expansion of any $x\in\mathbb{R}^+$ is constructed by dividing x by a suitable power β^k and finding $d_\beta(\frac{x}{\beta^k})=x_1x_2\cdots$ from which the β-expansion $\langle x\rangle_\beta$ is obtained, and moreover, $\langle x\rangle_\beta$ does not depend on the particular choice of exponent k. We derive the following observation, which leads us to a possible way how to properly and uniquely define $(-\beta,\ell)$-expansions of all real numbers.

Proposition 8. *Let $\beta>1$, $\ell\in(-1,0]$.*

1. *If $\ell\in\left(\frac{-\beta}{\beta+1},\frac{-1}{\beta+1}\right]$, then for any $z\in\mathbb{R}$ there exists exactly one admissible $(-\beta,\ell)$-representation.*
2. *If $\ell=\frac{-\beta}{\beta+1}$, then there exists a countable set of numbers $z\in\mathbb{R}$ with two distinct admissible $(-\beta,\ell)$-representations, first one using the string $d_{-\beta,\ell}(\ell)=l_1l_2l_3\cdots$ and second one using the string $1l_1l_2l_3\cdots$. For all other $z\in\mathbb{R}$, there exists exactly one admissible $(-\beta,\ell)$-representation.*
3. *If $\ell\notin\left[\frac{-\beta}{\beta+1},\frac{-1}{\beta+1}\right]$, then for uncountably many $z\in\mathbb{R}$ there exist at least two distinct admissible $(-\beta,\ell)$-representations.*

In all cases considered, i.e. if $\ell\in(-1,0]$, there exists for any $z\in\mathbb{R}$ exactly one admissible $(-\beta,\ell)$-representation $z_k\cdots z_0\bullet z_{-1}\cdots$ of z with $z_kz_{k-1}\cdots$ being strongly admissible.

Definition 9. *For any $x\in\mathbb{R}$, put $k:=\min\left\{i\ge 0 \;\middle|\; \left\{\frac{x}{(-\beta)^i},\frac{x}{(-\beta)^{i+1}}\right\}\subseteq[\ell,\ell+1)\right\}$. The* ***$(-\beta,\ell)$-expansion*** *of x, $\langle x\rangle_{-\beta,\ell}$, is defined by*

$$\langle x\rangle_{-\beta,\ell}:=\begin{cases} 0\bullet d_{-\beta,\ell}(x) & \text{if } k=0\,,\\ x_{k-1}\cdots x_0\bullet x_{-1}x_{-2}\cdots & \text{if } k\ge 1\,,\end{cases}$$

where $d_{-\beta,\ell}(x/(-\beta)^k)=x_{k-1}x_{k-2}x_{k-3}\cdots$.

Equivalently, $\langle x\rangle_{-\beta,\ell}$ is the admissible $(-\beta,\ell)$-representation $x_k\cdots x_0\bullet x_{-1}\cdots$ for which $x_kx_{k-1}\cdots$ is strongly admissible.

In analogy with the definition of β-expansions, we assign unique $\langle x\rangle_{-\beta}$ to each $x\in\mathbb{R}$ as possible leading zeros are omitted. If $\langle x\rangle_{-\beta}$ ends with suffix 0^ω, x is said to have **finite $(-\beta,\ell)$-expansion**.

4 Results on the $(-\beta)$-integers

With the proper definition of of $(-\beta,\ell)$-expansions for all real numbers at hand, we proceed to the definition of the set $\mathbb{Z}_{-\beta,\ell}$ of the so-called $(-\beta,\ell)$-integers. Our aim is to present analogous results to those of Thurston [Thu89] and Fabre [Fab95] for the set of β-integers, to describe the distances between consecutive $(-\beta,\ell)$-integers and give the formula for morphisms generating the encoding of $\mathbb{Z}_{-\beta,\ell}$ by an infinite word.

Definition 10. *The set of* $\boldsymbol{(-\beta,\ell)}$***-integers*** *(or just $(-\beta)$-integers) is defined by*

$$\mathbb{Z}_{-\beta}=\mathbb{Z}_{-\beta,\ell}:=\{x\in\mathbb{R}\mid\langle x\rangle_{-\beta,\ell}=x_l\cdots x_1x_0\bullet 0^\omega\}\,.$$

One can observe that the set $\mathbb{Z}_{-\beta,\ell}$ is nonempty if and only if $0\in\mathcal{A}_{-\beta,\ell}$, i.e. if and only if $\ell\in(-1,0]$. It is also self-similar ($-\beta\mathbb{Z}_{-\beta,\ell}\subseteq\mathbb{Z}_{-\beta,\ell}$) and it holds that $\mathbb{Z}_{-\beta,\ell}=\mathbb{Z}$ if and only if $\beta\in\mathbb{N}$.

A phenomenon unseen in Rényi numeration arises, there are cases when the set of $(-\beta,\ell)$-integers is trivial, i.e. when $\mathbb{Z}_{-\beta}=\{0\}$. This happens if and only if both numbers $\frac{1}{\beta}$ and $-\frac{1}{\beta}$ are outside of the interval $[\ell,\ell+1)$ or, equivalently, if and only if $\beta<-\frac{1}{\ell}$ and $\beta\le\frac{1}{\ell+1}$.

Let us recall the equivalent definition of the set of β-integers,

$$\mathbb{Z}_\beta^+=\bigcup_{i\ge0}\beta^iT_\beta^{-i}(0)\,.$$

In our general definition of $(-\beta,\ell)$-expansions with $\ell\in(-1,0]$, it holds only that

$$\mathbb{Z}_{-\beta,\ell}\subseteq\bigcup_{i\ge0}(-\beta)^iT^{-i}(0)$$

and the equality can only be guaranteed if $\ell\in\left[\frac{-\beta}{\beta+1},\frac{-1}{\beta+1}\right]$. Note that this directly follows from the relation between admissibility and strong admissibility of digit strings (see Proposition 8 and the following remark).

4.1 Distances between neighbors

In order to describe distances between adjacent $(-\beta,\ell)$-integers, we need some notation. Denote by $\mathcal{S}(k)$ the set of infinite $(-\beta,\ell)$-admissible digit strings such that erasing a prefix of length k yields 0^ω, i.e. for $k\ge0$, we have

$$\mathcal{S}(k)=\{a_{k-1}a_{k-2}\cdots a_00^\omega\mid a_{k-1}a_{k-2}\cdots a_00^\omega\text{ is }(-\beta,\ell)\text{-admissible}\}\,,$$

in particular $\mathcal{S}(0)=\{0^\omega\}$. Denote by $\mathrm{Max}(k)$ the string $a_{k-1}a_{k-2}\cdots a_00^\omega$ which is maximal in $\mathcal{S}(k)$ with respect to the alternate order and by $\max(k)$ its prefix of length k, so that $\mathrm{Max}(k)=\max(k)0^\omega$. Similarly, we define $\mathrm{Min}(k)$ and $\min(k)$. Thus,

$$\mathrm{Min}(k)\preceq_{\mathrm{alt}}r\preceq_{\mathrm{alt}}\mathrm{Max}(k)\,,\qquad\text{for all digit strings }r\in\mathcal{S}(k).$$

Let us define a "value function" γ. Consider a finite digit string $x_{k-1}\cdots x_1x_0$, then $\gamma(x_{k-1}\cdots x_1x_0)=\sum_{i=0}^{k-1}x_i(-\beta)^i$. With this notation we can give a theorem describing distances in $\mathbb{Z}_{-\beta,\ell}$ valid for the cases $\ell\in(-1,0]$.

Theorem 11. *Let $x<y$ be two consecutive $(-\beta,\ell)$-integers. Then there exist a finite string w over the alphabet $\mathcal{A}_{-\beta,\ell}$, a nonnegative integer $k\in\{0,1,2,\dots\}$ and digit $d\in\mathcal{A}_{-\beta,\ell}$ such that $w(d-1)\mathrm{Max}(k)$ and $wd\mathrm{Min}(k)$ are strongly $(-\beta,\ell)$-admissible strings and*

$$\begin{array}{lcll}x=\gamma(w(d-1)\max(k)) & < & y=\gamma(wd\min(k)) & \textit{for } k \textit{ even},\\ x=\gamma(wd\min(k)) & < & y=\gamma(w(d-1)\max(k)) & \textit{for } k \textit{ odd}.\end{array}$$

In particular, the distance $y - x$ between these $(-\beta, \ell)$-integers depends only on k and equals

$$\Delta'_k := \Big|(-\beta)^k + \gamma\big(\min(k)\big) - \gamma\big(\max(k)\big)\Big| . \tag{4}$$

Note that the above theorem does not give an explicit formula for distances between neighbors in $\mathbb{Z}_{-\beta,\ell}$ analogous to the one in [Thu89]. This is due to very tedious discussions necessary to describe the strings $\min(k), \max(k)$ in a completely general situation. Nevertheless, in the thesis we derive at least a recurrent formula for the extremal strings, valid for $\ell \in (-1, 0]$ under no additional assumptions on β.

We would like to emphasize two major distinctions between the sets of β- and $(-\beta, \ell)$-integers.

- ⋄ In contrast with $\mathbb{Z}^+_\beta$, where we have $\Delta_i \leq 1$ for all $i \geq 0$, i.e. all gaps between consecutive β-integers are of length at most 1, this bound does not in general hold for $\mathbb{Z}_{-\beta,\ell}$.
- ⋄ For any $\beta > 1$ it holds that $\mathbb{Z}^+_\beta \supsetneq \{0\}$ is an infinite set. As was already mentioned above, $\mathbb{Z}_{-\beta,\ell} \subseteq \bigcup_{i\geq 0}(-\beta)^i T^{-i}(0)$ and, moreover, it may happen that $\mathbb{Z}_{-\beta,\ell} = \{0\}$.

We derive explicit formulas for distances in $\mathbb{Z}_{-\beta,\ell}$ for Ito-Sadahiro case $\ell = \frac{-\beta}{\beta+1}$ and under some additional assumptions. Note that while this covers a large class of bases $-\beta$, some interesting examples are excluded, one may mention for instance the confluent Parry bases.

Theorem 12. *Assume $\ell = \frac{-\beta}{\beta+1}$ and set $m = \min\{k \in \mathbb{N} \mid d(\ell) = l_1 l_2 \cdots l_k 0^\omega\}$ if the minimum exists and $m = +\infty$ if $d(\ell)$ is not finite. If $0 < l_i$ and $l_1 > l_{2i}$ for all $i \leq m$, then the distances between adjacent $(-\beta, \ell)$-integers take values*

$$\Delta'_0 = 1 ,$$

$$\Delta'_k = \Big|(-1)^k + \sum_{i=1}^{\infty} \frac{l_{k-1+i} - l_{k+i}}{(-\beta)^i}\Big| , \quad k \in \{1, \ldots, m-1\} ,$$

$$\Delta'_m = \begin{cases} 1 - \frac{l_m}{\beta} & \text{for } m \text{ even} ,\\ \frac{l_m}{\beta} & \text{for } m \text{ odd} ,\end{cases}$$

$$\Delta'_{m+1} = \begin{cases} \Delta'_0 & \text{for } m \text{ even with } l_m < l_1 - 1 \text{ and for } m \text{ odd} ,\\ \Delta'_1 & \text{for } m \text{ even with } l_m = l_1 - 1 ,\end{cases}$$

$$\Delta'_k = \begin{cases} \Delta'_0 & \text{for odd } k \geq m+2 ,\\ \Delta'_1 & \text{for even } k \geq m+2 .\end{cases}$$

Moreover, all the distances are less than 2.

4.2 Encoding by infinite words

Let us now describe how we can code the set of $(-\beta, \ell)$-integers by an infinite word over the infinite alphabet $\mathbb{N}$.

Note that a similar study was performed by Steiner e.g. in [Ste12], where the antimorphisms generating encodings of $\mathbb{Z}_{-\beta,\ell}$ (for case $\ell = \frac{-\beta}{\beta+1}$) were obtained by a completely different approach, using the dynamical properties of $(-\beta)$-transformation $T_{-\beta}$.

Let $(z_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence satisfying

$$z_0 = 0 \quad \text{and} \quad \mathbb{Z}_{-\beta,\ell} = \{z_n \mid n \in \mathbb{Z}\}\,.$$

We define a bidirectional infinite word over an infinite alphabet $v_{-\beta} \in \mathbb{N}^{\mathbb{Z}}$, which codes the set of $(-\beta,\ell)$-integers. According to Theorem 11, for any $n \in \mathbb{Z}$ there exist a unique $k \in \mathbb{N}$, a word w with prefix 0 and a letter d such that

$$z_{n+1} - z_n = \big|\gamma(w(d-1)\max(k)) - \gamma(wd\min(k))\big|\,. \tag{5}$$

We define the word $v_{-\beta} = (v_i)_{i\in\mathbb{Z}}$ by $v_n = k$.

Theorem 13. *Let $v_{-\beta}$ be the word associated with $(-\beta,\ell)$-integers. There exists an antimorphism $\Phi : \mathbb{N}^* \to \mathbb{N}^*$ such that $\Psi = \Phi^2$ is a nonerasing nonidentical morphism and $\Psi(v_{-\beta}) = v_{-\beta}$. Φ is always of the form*

$$\Phi(k) = \begin{cases} S_k(k+1)\widetilde{R_k} & \text{for } k \text{ even,} \\ R_k(k+1)\widetilde{S_k} & \text{for } k \text{ odd}\,, \end{cases}$$

where $\widetilde{u}$ denotes the reversal of the word u and words R_j, S_j depend only on j and on $\min(k), \max(k)$ with $k \in \{j, j+1\}$.

Theorem 13 shows that a morphism over an infinite alphabet, fixing the word $v_{-\beta}$, exists for every β for which $v_{-\beta}$ can be defined. Such morphism can be explicitly described, whenever strings $\min(k)$ and $\max(k)$ are known. We further study under which conditions one can represent $(-\beta,\ell)$-integers by an infinite word over a restricted finite alphabet, so that it is still invariant under a primitive morphism.

Proposition 14. *Let v be an infinite word over the alphabet $\mathbb{N}$, and let $\Psi : \mathbb{N}^* \to \mathbb{N}^*$ be a morphism, such that $\Psi(v) = v$. Let Π be a letter-to-letter morphism $\Pi : \mathbb{N}^* \to \mathcal{B}^*$ which satisfies*

$$\Pi \circ \Psi = \Pi \circ \Psi \circ \Pi\,. \tag{6}$$

Then the infinite word $u = \Pi(v)$ is invariant under the morphism $\Pi \circ \Psi$.

Definition 15. *Let $v_{-\beta} \in \mathbb{N}^{\mathbb{Z}}$ be the infinite word encoding $\mathbb{Z}_{-\beta,\ell}$ and let there exist a letter-to-letter morphism $\Pi : \mathbb{N}^* \to \mathcal{B}^*$ from Proposition 14 with $\mathcal{B}$ finite of the form $\mathcal{B} = \{0, 1, \ldots, k\}$ chosen as minimal.*

We denote

$$u_{-\beta} := \Pi\big(v_{-\beta}\big) \quad \text{and} \quad \psi_{-\beta} = \Pi \circ \Phi\,.$$

By Proposition 14, $u_{-\beta}$ is a fixed point of antimorphism $\psi_{-\beta}$ (and of morphism $\psi_{-\beta}^2$).

We conclude this part by observing that a positive base analogue to our method from Theorem 13 yields an encoding of $\mathbb{Z}_\beta^+$ over infinite alphabet $\mathbb{N}$ for any (not necessarily Parry) base $\beta > 1$.

When deriving explicit formulas for antimorphisms generating the encoding of $\mathbb{Z}_{-\beta}$, we restrict ourselves to the Ito-Sadahiro case $\ell = \frac{-\beta}{\beta+1}$. In particular, we derive the antimorphisms generating $v_{-\beta}$ for β satisfying the assumptions of Theorem 12. The case $m = +\infty$, i.e. with infinite $d(\ell)$, is illustrated by the following theorem.

Theorem 16. *Assume* $\ell = \frac{-\beta}{\beta+1}$, *let the string* $d_{-\beta,\ell}(\ell) = l_1 l_2 l_3 \dots$ *satisfy* $0 < l_i$ *and* $l_1 > l_{2i}$ *for all* $i \geq 1$. *Then the antimorphism from Theorem 13 is of the form*

$$\begin{aligned}
\Phi(0) &= 0^{l_1-1}1\,, & \\
\Phi(2j) &= 0^{l_{2j+1}-1}(2j+1)0^{l_1-l_{2j}-1}1 & \text{for } j \geq 1,\\
\Phi(2j+1) &= 0^{l_{2j+1}-1}(2j+2)0^{l_1-l_{2j+2}-1}1 & \text{for } j \geq 0.
\end{aligned}$$

It holds for any β an Yrrap number and $\ell = \frac{-\beta}{\beta+1}$, the distances Δ'_k between consecutive $(-\beta,\ell)$-integers take only finitely many values and thus the set $\mathbb{Z}_{-\beta,\ell}$ can be coded by a bidirectional infinite word $u_{-\beta}$ over a finite alphabet $\mathcal{B} \subseteq \mathbb{N}$. By doing so for eventually periodic $d_{-\beta,\ell}(\ell) = l_1 l_2 \dots l_m (l_{m+1} \cdots l_{m+p})^\omega$, one finds that a prescription in terms of coefficients l_i cannot be written in one formula. Rather it differs dependently on whether the length of period is shorter or longer, even or odd. The discussion is tedious and in this thesis, we present explicit results just for some classes of numbers β with infinite periodic $d(\ell)$ together with the corresponding primitive morphisms fixing $u_{-\beta}$. The subcase of Theorem 12 with finite $d(\ell)$ is solved completely.

4.3 Spectral and combinatorial properties

In this thesis we demonstrate that the class of generalized multinacci numbers (i.e. confluent Parry numbers) β plays an exceptional role when it comes to studying spectra of $-\beta$. At the same time, it is exactly the class of bases for which the sets of β-integers and $(-\beta,\ell)$-integers (according to Ito-Sadahiro definition, i.e. for $\ell = \frac{-\beta}{\beta+1}$) are in certain sense similar. Let us recall that confluent Parry numbers are algebraic integers with minimal polynomials of the form

$$X^d - mX^{d-1} - \dots - mX - n\,, \quad \text{where } d \geq 1, m \geq n \geq 1\,.$$

Let us emphasize that the assumption $\ell = \frac{-\beta}{\beta+1}$ is maintained throughout this entire section. We recall the definition of spectrum of β, $X(\beta)$, and immediately follow with its negative base analogue, with the **spectrum of $-\beta$**, denoted by $X(-\beta)$:

$$X(\beta) := \Big\{ \sum_{j=0}^{N} a_j \beta^j \;\Big|\; N \in \mathbb{N}, a_j \in \mathcal{A}_\beta \Big\}\,, \quad X(-\beta) := \Big\{ \sum_{j=0}^{N} a_j (-\beta)^j \;\Big|\; N \in \mathbb{N}, a_j \in \mathcal{A}_{-\beta} \Big\}\,.$$

A natural question is: for given $\beta > 1$, are the sets $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ similar in any way? From our point of view, the “similarity” can be expressed by three properties (ordered in such a way that each one implies all of the previous):

1. both $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ contain only distances of length ≤ 1
2. the sets of distances in $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ are the same
3. infinite words encoding $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ have the same language (which is guaranteed e.g. in case when morphisms fixing them are conjugated)

Let us recall that two morphisms π, ρ over an alphabet $\mathcal{A}$ are **conjugated** if there exists a word $w \in \mathcal{A}^*$ such that either

$$w\pi(a) = \rho(a)w \;\text{ for all } a \in \mathcal{A}\,, \;\text{ or } \pi(a)w = w\rho(a) \;\text{ for all } a \in \mathcal{A}\,.$$

Note that for the encoding of $\mathbb{Z}_\beta^+$ (denoted by u_β or v_β, depending on the finiteness of its alphabet) there always exists a morphism fixing it. Similarly for the encoding $u_{-\beta}$ (or $v_{-\beta}$) of $\mathbb{Z}_{-\beta}$ there exists an antimorphism fixing it. Although it does not make sense to decide whether a morphism φ is conjugated to an antimorphism ψ, it suffices to consider their second iterations φ^2 and ψ^2 as both are morphisms.

We present a negative base analogue to the result of Frougny [Fro92] by stating that the spectrum $X(-\beta)$ coincides with the set of $(-\beta)$-expansions if and only if β belongs to a subclass of confluent Parry numbers. Moreover, exactly in those cases the sets of β- and $(-\beta)$-integers are encoded by fixed points of conjugate morphisms.

Theorem 17. *Let $\beta > 1$. Denote by φ_β the canonical morphism of β (over a finite or an infinite alphabet) and $\psi_{-\beta}$ the antimorphism fixing the infinite word coding $\mathbb{Z}_{-\beta}$. Then the following conditions are equivalent:*

1. *β is a confluent Parry number with minimal polynomial $X^d - mX^{d-1} - \cdots - mX - n$ with $m \geq n \geq 1$, such that $n = m$ for d even.*
2. *φ_β^2 is conjugated with $\psi_{-\beta}^2$.*
3. *$\mathbb{Z}_{-\beta} = X(-\beta)$.*

Note that the equivalence $1 \Leftrightarrow 3$ is not an exact analogue to the result of [Fro92] as not all confluent Parry numbers satisfy the theorem above. This missing class of confluent Parry numbers is the subject of the following remark.

Remark 18. *Let β be a confluent Parry number of even degree $d \geq 2$ with minimal polynomial $p(X) = X^d - m(X^{d-1} + \ldots + X) - n$, $m > n \geq 1$. The sets of distances in $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ do not coincide, hence φ_β^2 and $\psi_{-\beta}^2$ are not conjugated. Nevertheless, certain level of similarity can still be found. Recall that $u_{-\beta}$ is an infinite word coding $\mathbb{Z}_{-\beta}$. If we apply a morphism $\pi(i) : \{0, \ldots, d-1\}^* \to \{0, \ldots, d-1\}^*$ on $u_{-\beta}$, where*

$$\pi(i) = \begin{cases} i & \text{if } i \in \{0, \ldots, d-2\}, \\ 0(d-1) & \text{if } i = d-1, \end{cases}$$

then one easily verifies that the words u_β and $\pi(u_{-\beta})$ have the same language.

5 Results on the generalization of the unit sum number problem

In this thesis, we study possible generalizations of known results on representing algebraic integers by units and also on a related topic, on bounding the lengths of arithmetic progressions of algebraic integers. As finite sums of units can be actually viewed as linear combinations of algebraic integers with norm ≤ 1 in modulus, with bounded integer coefficients, two straightforward generalizations follow. We can consider

- finite sums of algebraic integers with norm in modulus $\leq m$ when $m > 0$ (studied in Subsection 5.2), or
- linear combinations of units with coefficients coming from some given subset of $\mathbb{Q}$ (see Subsection 5.3).

5.1 Arithmetic progressions of algebraic integers

Denote by O_K the ring of algebraic integers in a field K and for $m > 0$ put

$$\mathcal{N}_m^* := \{\gamma \in O_K \mid |N(\gamma)| \leq m\},$$

and write

$$t \times \mathcal{N}_m^* := \{\gamma_1 + \cdots + \gamma_t \mid \gamma_i \in \mathcal{N}_m^* \ (i \in \{1, \ldots, t\})\},$$

where t is a positive integer. We prove a generalization of the results of Newman [New90] and Bérczes et al. [BHP10].

Theorem 19. *Let $K = \mathbb{Q}(\alpha)$, $m > 0$ and $t > 0$. The length of any nonconstant arithmetic progression in $t \times \mathcal{N}_m^*$ is at most $c_1(m, t, d, D(K))$, where $c_1(m, t, d, D(K))$ is an explicitly computable constant depending only on m, t, and on the degree d and discriminant $D(K)$ of K.*

5.2 First generalization - sums of elements of small norm

Jarden and Narkiewicz [JN07] proved that $u(O_K) \geq \omega$ for any number field K. To formulate our next result, we extend the notion of unit sum number, in case that given ring R is the ring of integers in some number field K.

Definition 20. *Let $K = \mathbb{Q}(\alpha)$ with the ring of integers O_K. The* ***m-norm sum number*** $\boldsymbol{u_m(O_K)}$ *of O_K is defined by*

$$u_m(O_K) := \begin{cases} t \in \mathbb{N} & \text{if } t \text{ is minimal such that every } \gamma \in O_K \text{ is a sum of} \\ & \quad \text{at most } t \text{ elements of } \mathcal{N}_m^*, \\ \omega & \text{if every } \gamma \in O_K \text{ is a sum of elements of } \mathcal{N}_m^* \text{ but} \\ & \quad \text{there is no bound for the number of summands,} \\ \infty & \text{if some } \gamma \in O_K \text{ is not a sum of elements of } \mathcal{N}_m^*. \end{cases} \tag{7}$$

Hence, instead of sums of units we consider sums of integers of bounded norm. Clearly, $u(O_K) = u_1(O_K)$ holds.

Theorem 21. *Let $K = \mathbb{Q}(\alpha)$ be and algebraic number field and let $m > 0$. It holds that $u_m(O_K) \geq \omega$, i.e. for every $m, t \in \mathbb{N}$ there exists $\gamma \in O_K$ which cannot be obtained as a sum of at most t terms from $\mathcal{N}_m^*$.*

It is known that for infinitely many number fields K we have $u(O_K) = \infty$. In contrast to this result, the next theorem shows that $u_m(O_K) = \omega$ is always valid if m is “large enough” with respect to the discriminant and the degree of K.

Theorem 22. *For every algebraic number field $K = \mathbb{Q}(\alpha)$ there exists a positive integer $m_0 = m_0(D(K), d)$ depending only on the discriminant and the degree of K, such that for any $m \geq m_0$ we have $u_m(O_K) = \omega$, i.e. any $\gamma \in O_K$ can be obtained as a sum of elements from $\mathcal{N}_m^*$.*

5.3 Second generalization - linear combinations of units

At this point let us recall that the field K is called a CM-field, if it is a totally imaginary quadratic extension of a totally real number field.

Theorem 23. *Suppose that either K is not a CM-field, or K is a CM-field containing a root of unity different from ± 1. Then there exists a positive integer $\ell = e^{c_6(d)R(K)}$ where $c_6(d)$ is a constant depending only on the degree of K, such that any $\gamma \in O_K$ can be obtained as a linear combination of (not necessarily distinct) units of K with coefficients $\{1, 1/2, 1/3, \ldots, 1/\ell\}$.*

6 Results on quartic DUG-fields

In the following, let ζ_μ denote a primitive μ-th root of unity. Hajdu and Ziegler in [HZ14] provided the following list of number fields which contain all candidates for distinct unit generated (DUG) fields among totally complex quartic ones.

- $\mathbb{Q}(\zeta_\mu)$ where $\mu = 5, 8, 12$ or,
- $\mathbb{Q}(\gamma)$ where γ is the root of one of the polynomials $X^4 - X + 1$, $X^4 + X^2 - X + 1$, $X^4 + 2X^2 - 2X + 1^\dagger$, $X^4 - X^3 + X + 1^\ddagger$, $X^4 - X^3 + X^2 + X + 1^\ddagger$, $X^4 - X^3 + 2X^2 - X + 2^\dagger$ or,
- $\mathbb{Q}(\sqrt{a + b\zeta_4})$, with $(a, b) = (1, 1), (1, 2), (1, 4), (7, 4)^\dagger$ or,
- $\mathbb{Q}(\sqrt{a + b\zeta_3})$, with $(a, b) = (2, 1)$, $(4, 1)$, $(8, 1)$, $(3, 2)$, $(4, 3)$, $(7, 3)$, $(11, 3)$, $(5, 4)$, $(9, 4)$, $(13, 4)$, $(12, 5)$, $(11, 7)$, $(9, 8)$, $(15, 11)$, $(19, 11)^\dagger$, $(17, 12)^\dagger$, $(17, 16)^\dagger$ or,
- $\mathbb{Q}(\zeta_4, \sqrt{5})$ or $\mathbb{Q}(\zeta_3, \sqrt{d})$, with $d = 5, 6, 21$ or,
- $\mathbb{Q}\left(\sqrt{-1 - \sqrt{2}}\right)$ or $\mathbb{Q}\left(\sqrt{-\frac{1+\sqrt{5}}{2}}\right)$.

Table 1: Candidates for totally complex quartic DUG fields. Markers † and ‡ are necessary for the statement of Theorem 24.

The authors of [HZ14] only managed to show that the fields $K = \mathbb{Q}(\gamma)$, where

$$\gamma \in \left\{\alpha, \zeta_5, \zeta_8, \zeta_{12}, \sqrt{-1 - \sqrt{2}}, \sqrt{-\tfrac{1+\sqrt{5}}{2}}, \zeta_3 + \sqrt{5}, \zeta_4 + \sqrt{5}\right\}$$

and α is a root of the polynomial $X^4 + X^2 - X + 1$ are indeed DUG. They also conjectured that all the remaining fields in Table 1 are DUG as well.

We extend the method of Thuswaldner and Ziegler from [TZ11] to totally complex number fields and apply this method to extend the list of DUG-fields, i.e. those fields K with their unit sum height $\omega(K)$ equal to one. Unfortunately we failed in proving that all fields listed in Table 1 are distinct unit generated, but at least we can provide upper bounds for the unit sum height.

Theorem 24. *If K is a totally complex quartic field of the list in Table 1, then $\omega(K) \leq 3$. Moreover all such fields are DUG except those marked with † or ‡. Those fields marked with † satisfy at least $\omega(K) \leq 2$ and those marked with ‡ satisfy only $\omega(K) \leq 3$.*

In the following we generalize a theorem from [TZ11] to the case which includes totally complex number fields. The real Pisot number is replaced by the notion of **complex Pisot number**, i.e. a nonreal algebraic integer α with $|\alpha| > 1$ such that the remaining conjugates other than $\overline{\alpha}$ are less than one in modulus.

With this generalization at hand, we consider the case that a totally complex number field K contains a primitive μ-th root of unity with $\mu > 2$. This enables us to prove Theorem 24 up to the second item in the list of fields given there (see Subsection 6.2). In Section 6.3, we apply a variant of our method to the remaining fields and prove Theorem 24 up to the case that $K = \mathbb{Q}(\gamma)$, where γ is a root of $X^4 - X + 1$. This special case is solved in Subsection 6.4 independently, by a combinatorial approach.

As pointed out already in [TZ11], the problem of determining $\omega(K)$ is connected to nonstandard positional representation of numbers. Consider a field K of unit rank 1, which covers, besides totally real quadratic and not totally real cubic fields also totally complex quartic fields which are subject of this paper. By Dirichlet's theorem, all units in K are of the form $\zeta_\mu^i \epsilon^j$, $i, j \in \mathbb{Z}$, where ϵ is the fundamental unit and ζ_μ^i for $1 \leq i \leq \mu$ form a finite set of all roots of unity in K.

The fact that $\omega(K) \leq w$ can be rephrased by saying that every element of O_K can be represented as $\sum_{j=l}^{k} a_j \epsilon^j$, where the 'digits' a_j take values in the finite set

$$\Sigma = \Sigma_\mu(w) := \Big\{ \sum_{i=1}^{\mu} d_i \zeta_\mu^i \;\Big|\; 0 \leq d_i \leq w \;\text{ for }\; 1 \leq i \leq \mu \Big\} . \tag{8}$$

Assume that the fundamental unit ϵ also generates the integral basis of the ring of integers in K, i.e. $O_K = \mathbb{Z}[\epsilon]$. Then the question reformulates to asking whether the set of numbers with finite expansion in base ϵ with digits in Σ satisfies

$$\mathrm{Fin}_\Sigma(\epsilon) := \Big\{ \sum_{i=l}^{k} a_i \epsilon^i \;\Big|\; k, l \in \mathbb{Z},\; a_i \in \Sigma \Big\} = \mathbb{Z}[\epsilon, \epsilon^{-1}] = \mathbb{Z}[\epsilon] , \tag{9}$$

which will be true, if $\mathrm{Fin}_\Sigma(\epsilon)$ is closed under addition. This is a generalization of the so-called finiteness property studied in numeration systems, first introduced for Rényi β-expansions of real numbers by Frougny and Solomyak [FS92].

6.1 Determining the upper bounds for the unit sum length

Let K be a number field with the ring of integers O_K of degree $d = s + 2t$, where, as usual, s and $2t$ denote the number of real and nonreal conjugates of the field primitive element, respectively. Also let us fix real field isomorphisms $\sigma_1, \ldots, \sigma_s$ and the complex ones $\sigma_{s+1} = \bar{\sigma}_{s+t+1}, \ldots, \sigma_{s+t} = \bar{\sigma}_{s+2t}$ of K. For $\alpha \in K$ we denote by $\alpha^{(i)} = \sigma_i(\alpha)$ the conjugates of α and by convention we write $\alpha = \alpha^{(s+1)}$ (for in the following we consider nonreal fields K)

Let $\epsilon \in O_K$ be a complex Pisot number, i.e. such that $|\epsilon| > 1$ and $|\epsilon^{(i)}| < 1$ for all $i = 1, \ldots, s+t$, $i \neq s+1$. Given a finite set $\Sigma \subset O_K$, denote

$$C_i := \max\{|c^{(i)}| \mid c \in \Sigma\} , \qquad \text{for } i \in \{1, \ldots, s+t\},\; i \neq s+1.$$

Consider a compact set $P \subset \mathbb{C}$, containing at least a neighborhood of 0 and denote by $\mathcal{B}(\epsilon, \Sigma, P)$ the finite set called **cylinder**, defined by

$$\mathcal{B}(\epsilon, \Sigma, P) := \Big\{ \alpha \in O_K \;\Big|\; \alpha \in P \text{ and } |\alpha^{(i)}| \leq \frac{C_i}{1 - |\epsilon^{(i)}|} \text{ for } i \in \{1, \ldots, s+t\} , i \neq s+1 \Big\} .$$

Theorem 25. *Let $\epsilon \in O_K$ be a complex Pisot number. With the notation above, assume that*

$$\epsilon P \subset \bigcup_{a\in\Sigma}(a+P)\,. \tag{10}$$

Then for each $\alpha \in O_K$ there exist $N, n \in \mathbb{N}$ such that

$$\alpha\epsilon^N = \beta + \sum_{i=0}^{n} c_i\epsilon^i, \tag{11}$$

with $c_i \in \Sigma$ and β is contained in the finite set $\mathcal{B}(\epsilon,\Sigma,P)$. The elements of $\mathcal{B}(\epsilon,\Sigma,P)\setminus\{0\}$ will be called ***critical points****.*

6.2 Application to fields with nontrivial roots of unity

Assume that K contains a μ-th root of unity with $\mu > 2$ and denote by ζ_μ some primitive μ-th root of unity. The digit set Σ will be taken, as in (8), by the set of all possible sums of roots of unity with bounded coefficients.

First, let us assume that K is a complex (not necessarily quartic) field that contains a fourth root of unity $\zeta_4 = \imath$. Given a complex Pisot number $\epsilon \in O_K$, we apply Theorem 25 to the case where $P \subset \mathbb{C}$ is the square with vertices $\frac{\pm1\pm\imath}{2}$ and obtain a simple criterion such that the covering property (10) holds:

Lemma 26. *Let $P \subset \mathbb{C}$ be the square with vertices $\frac{\pm1\pm\imath}{2}$. Let $\eta = \epsilon\frac{1+\imath}{2}$, then (10) is satisfied, provided*

$$\max\{|\Re(\eta)|, |\Im(\eta)|\} \le \frac{1+2w}{2}\,.$$

We may then apply Lemma 26 together with Theorem 25 to the fields $\mathbb{Q}(\sqrt{1+\zeta_4})$, $\mathbb{Q}(\sqrt{1+2\zeta_4})$, $\mathbb{Q}(\sqrt{1+4\zeta_4})$ and $\mathbb{Q}(\sqrt{7+4\zeta_4})$. For the complex Pisot number ϵ we take the fundamental unit of K. Indeed, the fundamental unit ϵ can be chosen in such a way that $|\epsilon| = |\bar{\epsilon}| > 1$. Moreover, as ϵ is a unit, i.e. $|N(\epsilon)| = 1$, it immediately follows that the remaining two conjugates are in modulus less than 1. By a computer search we were able to confirm that all critical points can be written in the form $\sum_{k=-1}^{-B} s_i\epsilon^i$ with $s_i \in \Sigma_4(w)$.

Now, let us assume that K is a complex (not necessarily quartic) field that contains sixth roots of unity. In this case we choose P to be a regular hexagon. Again, ϵ is the fundamental unit, which can be chosen to be a complex Pisot number. We derive a similar covering property criterion to Lemma 26, which then, in an analogous way, leads to the solution in case of fields $\mathbb{Q}(\sqrt{a+b\zeta_3})$, with $(a,b) = (2,1)$, $(4,1)$, $(8,1)$, $(3,2)$, $(4,3)$, $(7,3)$, $(11,3)$, $(5,4)$, $(9,4)$, $(13,4)$, $(12,5)$, $(11,7)$, $(9,8)$, $(15,11)$, $(19,11)$, $(17,12)$, $(17,16)$ and $\mathbb{Q}(\zeta_3,\sqrt{d})$, with $d = 6, 21$.

6.3 Application to five special cases

Now we consider the remaining five number fields in Theorem 24, namely those which do not contain any roots of unity ζ_μ for $\mu > 2$. The same approach as in the previous section will not lead to success, since the alphabet $\Sigma_2(w) = \{-w,\dots,0,\dots,w\}$ is contained in the real line.

Instead, we take the digit set

$$\Sigma = \{d_0 + d_1\tilde{\epsilon} \mid -w \le d_0, d_1 \le w\}\ ,$$

and expand the number $\alpha \in O_K$ in base $\epsilon = \tilde{\epsilon}^2$, where $\tilde{\epsilon}$ is a fundamental unit with $|\tilde{\epsilon}| > 1$. The compact set $P \subset \mathbb{C}$ is taken to be the parallelogram with vertices $\frac{\pm 1 \pm \tilde{\epsilon}}{2}$. Again, a covering property criterion is derived, leading to the results for fields $K = \mathbb{Q}(\gamma)$ with γ having minimal polynomial $X^4 - X^3 + X^2 + X + 1$, $X^4 - X^3 + X + 1$, $X^4 + 2X^2 - 2X + 1$, $X^4 - X^3 + 2X^2 - X + 2$ and $X^4 - X + 1$ (which is further improved in the next subsection).

6.4 Combinatorial approach

By using different means, we prove that $K = \mathbb{Q}(\gamma)$, where γ is a root of the polynomial $X^4 - X + 1$, is DUG. Although we already proved in the previous section that $\omega(K) \leq 2$ we do not assume this result in this section.

Proposition 27. *The field $K = \mathbb{Q}(\gamma)$ with γ being a root of the polynomial $X^4 - X + 1$ is DUG.*

In this approach, we use an equivalent formulation of DUG property. Since $\{1, \gamma, \gamma^2, \gamma^3\}$ is an integral basis of the ring of integers O_K, we have $O_K = \mathbb{Z}[\gamma] = \mathbb{Z}[\gamma, \gamma^{-1}]$ and we can write every element $\alpha \in O_K$ in the form $\alpha = \sum_{n=-\infty}^{\infty} v_n \gamma^n$ with $v_n \in \mathbb{Z}$ and $v_n \neq 0$ for at most finitely many indices. Such a γ-representation of α can be written as

$$\alpha = \cdots v_2 v_1 v_0 \bullet v_{-1} v_{-2} \cdots ,$$

where the fractional point $\bullet$ separates between the coefficients at negative and nonnegative powers of the base γ. We are only interested in the fact that nonvanishing coefficients in the γ-representation are finitely many. Thus we will abbreviate representation above by the finite word $v_k v_{k-1} \cdots v_{\ell+1} v_\ell$, where the indices k and ℓ are such that $v_n = 0$ for all $n > k$ and all $n < \ell$, without marking the fractional point. Note that the γ-representation is not unique. Since $\gamma^n(\gamma^4 - \gamma + 1) = 0$ for all n, position-wise addition or subtraction of $100\bar{1}1$, with $\bar{1} = -1$, at any position does not change the value of α but only its γ-representation, i.e. the words $v_k \cdots v_n v_{n-1} v_{n-2} v_{n-3} v_{n-4} \cdots v_\ell$ and $v_k \cdots (v_n + 1) v_{n-1} v_{n-2} (v_{n-3} - 1)(v_{n-4} + 1) \cdots v_\ell$ represent the same element $\alpha \in O_K$. This application of $w = 100\bar{1}1$ is called the **rewriting rule**.

From this point of view, any element $\alpha \in O_K$ has some γ-representation

$$x_3 x_2 x_1 x_0 , \quad x_i \in \mathbb{Z} . \tag{12}$$

Moreover, since γ is the fundamental unit of $K = \mathbb{Q}(\gamma)$ and there are no nonreal roots of unity in K, we have

$$U_K = \{\pm \gamma^k \mid k \in \mathbb{Z}\} .$$

Consequently, if $\alpha \in O_K$ is also a sum of distinct units, there exists another γ-representation of the form

$$v_k v_{k-1} \cdots v_0 v_{-1} \cdots v_\ell , \quad v_i \in \{\bar{1}, 0, 1\}, \ell, k \in \mathbb{Z} . \tag{13}$$

Hence, if we want to prove that the field K is DUG, we have to show that any representation of the form (12) can be rewritten into (13) without changing the value of the represented number.

Let us note that the method used in the proof of Proposition 27 is very particular for the field $K = \mathbb{Q}(\gamma)$, where γ is a root of the polynomial $X^4 - X + 1$, which provided us rewriting rules w with low weight but large support. We failed in proving an analogous result for the remaining cases of Theorem 24, since the corresponding fields seem not to provide such rewriting rules.

The possibility to rewrite any finite word with integer digits into the alphabet $\Sigma = \{-1, 0, 1\}$ is closely connected to the finiteness property (9) of numeration systems. Being in general a highly nontrivial problem, only few results are known. For example, in [FPS11], it was shown that for any algebraic integer γ without conjugates on the unit circle there exists an alphabet Σ of consecutive integers, such that $\mathrm{Fin}_\Sigma(\gamma)$ is closed under addition. This is however very far from stating that $\Sigma = \{-1, 0, 1\}$ is sufficient.

7 Conclusions

Our results on $(-\beta)$-expansions can serve as an illustration that although numerous analogies between expansions in positive and negative base actually hold, negative base results are often much more complicated and technical, when compared to their positive base counterparts. Nevertheless it makes sense to study $(-\beta)$-expansions further. For instance, in a positive base, the fact that the expansions of negative numbers are "artificially" defined only by prefixing a minus sign to the expansions of positive numbers may be inconvenient. Consequently, the encoding of the whole set $\mathbb{Z}_\beta$ cannot be generated by a single morphism. This phenomenon is eliminated by using a negative base.

The utility of systems with negative base was confirmed also in the study of generalized spectra of numbers. Also arithmetical aspects of negative bases may sometimes appear easier than in the case of positive bases, see for instance [FPS13]. Deeper knowledge of $(-\beta)$-expansions in general can extend the repertoire for already known applications, for instance in case of the so-called β-encoders.

Considering how much is already known about Rényi β-expansions, there still remain numerous properties, phenomena and analogies yet to uncover about $(-\beta)$-expansions, let alone about any further generalizations. For instance, we used a parameter $\ell = \ell(\beta)$ and studied $(-\beta, \ell)$-expansions of numbers from $[\ell, \ell + 1)$ for various values of the base $-\beta$. Instead of this approach, it could be interesting to proceed the other way around, i.e. to fix the base $-\beta$ and then study how various objects, such as the reference strings in the admissibility criterion, the set $Fin(-\beta)$ of numbers with finite expansions or the so-called $(-\beta)$-shift, are influenced by changing the value of ℓ. As the finiteness property has already been studied in the Ito-Sadahiro case for quadratic and some cubic Pisot bases, it makes sense to further study for instance the length of the fractional part arising when adding, subtracting or multiplying two $(-\beta)$-integers.

To our knowledge, little attention has been so far dedicated to the generalization of results about the so-called univoque numbers and unique expansions in negative bases. For β-expansions, this topic was introduced in [EJK90] and further studied in many works.

One can view both β- and $(-\beta)$-expansions as particular cases of positional representation in a base $\alpha \in \mathbb{C}$ with digits coming from an alphabet $\mathcal{A} \subset \mathbb{C}$. Various approaches have been chosen in this field so far, let us mention for instance the study of expansions in a real base defined by a generalization of the β-transformation e.g. in [KS12] and [Gór07]. From other interesting approaches, one can mention canonical number systems (CNS) or shift radix systems (SRS).

As we demonstrated, several number-theoretical problems as the DUG property of number fields, hence indirectly also the unit sum number problem, are connected to the arithmetic properties in positional numeration.

Concerning the DUG property, two distinct approaches were applied to obtain the results in this work. The geometrical one was based on an approximation of arbitrary algebraic integer by its representation in base equal to the fundamental unit of the field with the error term coming from a bounded set $P \subset \mathbb{C}$, while the combinatorial one rephrased the problem in terms of rewriting representations of algebraic integers into representations of another type. The geometrical approach assumed P to be a square, a parallelogram or a hexagon. One could study whether a choice of a different shape (maybe a self-similar and fractal one) further improves our recent results. Similarly, although the combinatorial approach using the rewriting rules seemed too particular for the one field considered, it would be useful to discover a modification which would provide good results also for the other fields from the given list of candidates.

Nevertheless, in general, both the unit sum number problem and the DUG property themselves are far from the complete solution as all the available results providing classification of number fields with respect to the unit sum number and the unit sum height are limited to only low degree fields, namely those with unit rank 1. There are definitely opportunities for further research and it would be remarkable if more connections similar to those presented in this thesis were found in the future.

Author's publications

The thesis is based upon the following articles.

[I] Daniel Dombek, Zuzana Masáková, and Edita Pelantová. Number representation using generalized $(-\beta)$-transformation. *Theoret. Comput. Sci.*, 412(48):6653–6665, 2011.

[II] Petr Ambrož, Daniel Dombek, Zuzana Masáková, and Edita Pelantová. Numbers with integer expansion in the numeration system with negative base. *Funct. Approx. Comment. Math.*, 47(part 2):241–266, 2012.

[III] Daniel Dombek. Substitutions over infinite alphabet generating $(-\beta)$-integers. *Internat. J. Found. Comput. Sci.*, 23(8):1627–1639, 2012.

[IV] Daniel Dombek. Generating $(\pm\beta)$-integers by conjugated morphisms. In *Local Proceedings of WORDS 2013*, volume 20 of *TUCS Lecture Notes*, pages 14–25. Turku Centre for Computer Science, Turku, 2013.

[V] Daniel Dombek, Lajos Hajdu, and Attila Pethő. Representing algebraic integers as linear combinations of units. To appear in *Period. Math. Hungar.*, 2014. doi:10.1007/s10998-014-0020-9

[VI] Daniel Dombek, Zuzana Masáková, and Tomáš Vávra. Confluent Parry numbers, their spectra, and integers in positive- and negative-base number systems. Submitted to *J. Théor. Nombres Bordeaux*, 2014.

[VII] Daniel Dombek, Zuzana Masáková, and Volker Ziegler. Distinct unit generated totally complex quartic fields. Conditionally accepted to *J. Number Theor.*, 2014.

Note on authorship

In the following, the author wishes to specify his particular contributions.

- ◇ Papers [I] and [II] are results of a joint work, where it is difficult to precisely separate individual contributions.
- ◇ In paper [VI], the author's personal contribution covers the results about similarity of the sets of β- and $(-\beta)$-integers.
- ◇ Among results of [V], the author contributed mainly to the parts generalizing the unit sum number problem.
- ◇ Among results cited from [VII], the author's personal contribution covers the combinatorial proof of DUG property. In particular, the actual computations in remaining parts were performed by other coauthors.
- ◇ Papers [III] and [IV] are the author's sole contributions, although related to others. In particular, article [III] generalizes the results of [II] into a more general context of [I]. Publication [IV], containing the author's original results in shortened form, was subsequently extended and complemented by related results into [VI].

All coauthors have agreed to include the joint results as parts of this thesis.

Author's citations

The following articles cite a publication mentioned in the previous list. Autocitations are excluded.

1. Karma Dajani and Sanjay D. Ramawadh. Symbolic dynamics of $(-\beta)$-expansions. *J. Integer Seq.*, 15(2):Article 12.2.6, 21, 2012.

2. Christiane Frougny and Anna Chiara Lai. Negative bases and automata. *Discrete Math. Theor. Comput. Sci.*, 13(1):75–93, 2011.

3. Charlene Kalle. Isomorphisms between positive and negative β-trans-formations. *Ergodic Theory Dynam. Systems*, 34(1):153–170, 2014.

4. Tohru Kohda, Yoshihiko Horio, Yoichiro Takahashi, and Kazuyuki Aihara. Beta encoders: symbolic dynamics and electronic implementation. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 22(9):1230031, 55, 2012.

5. Zuzana Masáková and Edita Pelantová. Purely periodic expansions in systems with negative base. *Acta Math. Hungar.*, 139(3):208–227, 2013.

6. Wolfgang Steiner. On the Delone property of $(-\beta)$-integers. In *Proceedings of WORDS*, pages 247–256, 2011.

7. Wolfgang Steiner. On the structure of $(-\beta)$-integers. *RAIRO Theor. Inform. Appl.*, 46(1):181–200, 2012.

8. Wolfgang Steiner. Digital expansions with negative real bases. *Acta Math. Hungar.*, 139(1-2):106–119, 2013.

Author's conference talks and other lectures

1. *On distinct unit generated fields that are totally complex*, Numeration and Substitution 2014, Debrecen (Hungary)

2. *On the characterization of all β with $(\pm\beta)$-integers encoded by conjugated morphisms*, WORDS 2013, Turku (Finland)

3. *On the characterization of all β with $(\pm\beta)$-integers encoded by conjugated morphisms*, 28th Journées Arithmétiques, 2013, Grenoble (France)

4. *On the generalizations of the unit sum number problem* (invited lecture), Graz University of Technology, November 2012, Graz (Austria)

5. *Integers in numeration systems with irrational bases*, Česko-Slovenská MELA - Meeting on Languages, 2012, Telč (Czech Republic)

6. *Algebraic integers as combinations of units* (poster), School on Combinatorics, Automata and Number Theory CANT 2012, Marseille (France)

7. *Numeration systems with negative base* (invited lecture), Mathematical Institute, University of Debrecen, October 2011, Debrecen (Hungary)

8. *Substitutions over infinite alphabet generating $(-\beta)$-integers*, WORDS 2011, Prague (Czech Republic)

9. *Numeration systems with negative base*, 27th Journées Arithmétiques, 2011, Vilnius (Lithuania)

10. *Expansions of numbers in a negative base*, Numération 2011, Liège (Belgium),

11. *Generalised numeration systems with negative base*, Analytic and algebraic methods VIII, 2011, Prague (Czech Republic)

12. *Numeration systems with negative base* (invited lecture), Faculty of Informatics, University of Debrecen, September 2010, Debrecen (Hungary)

13. *Numbers with integer expansion in the numeration system with negative base*, Analytic and algebraic methods VII, 2010, Prague (Czech Republic)

References

[AS07] Shigeki Akiyama and Klaus Scheicher. Symmetric shift radix systems and finite expansions. *Math. Pannon.*, 18(1):101–124, 2007.

[AV05] Nahid Ashrafi and Peter Vámos. On the unit sum number of some rings. *Q. J. Math.*, 56(1):1–12, 2005.

[Bel76] Paul Belcher. A test for integers being sums of distinct units applied to cubic fields. *J. London Math. Soc. (2)*, 12(2):141–148, 1975/76.

[BFGK98] Čestmír Burdík, Christiane Frougny, Jean-Pierre Gazeau, and Rudolf Krejcar. Beta-integers as natural counting systems for quasicrystals. *J. Phys. A*, 31(30):6449–6472, 1998.

[BFT11] Fabrizio Barroero, Christopher Frei, and Robert F. Tichy. Additive unit representations in rings over global fields—a survey. *Publ. Math. Debrecen*, 79(3-4):291–307, 2011.

[BHP10] Attila Bérczes, Lajos Hajdu, and Attila Pethő. Arithmetic progressions in the solution sets of norm form equations. *Rocky Mountain J. Math.*, 40(2):383–395, 2010.

[EJK90] Pál Erdős, István Joó, and Vilmos Komornik. Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems. *Bull. Soc. Math. France*, 118(3):377–390, 1990.

[Fab95] Stéphane Fabre. Substitutions et β-systèmes de numération. *Theoret. Comput. Sci.*, 137(2):219–236, 1995.

[FL11] Christiane Frougny and Anna Chiara Lai. Negative bases and automata. *Discrete Math. Theor. Comput. Sci.*, 13(1):75–93, 2011.

[FPS11] Christiane Frougny, Edita Pelantová, and Milena Svobodová. Parallel addition in non-standard numeration systems. *Theoret. Comput. Sci.*, 412(41):5714–5727, 2011.

[FPS13] Christiane Frougny, Edita Pelantová, and Milena Svobodová. Minimal digit sets for parallel addition in non-standard numeration systems. *J. Integer Seq.*, 16(2):Article 13.2.17, 36, 2013.

[Fro92] Christiane Frougny. Confluent linear numeration systems. *Theoret. Comput. Sci.*, 106(2):183–219, 1992.

[FS92] Christiane Frougny and Boris Solomyak. Finite beta-expansions. *Ergodic Theory Dynam. Systems*, 12(4):713–723, 1992.

[Gór07] Paweł Góra. Invariant densities for generalized β-maps. *Ergodic Theory Dynam. Systems*, 27(5):1583–1598, 2007.

[HZ14] Lajos Hajdu and Volker Ziegler. Distinct unit generated totally complex quartic fields. *Math. Comp.*, 83(287):1495–1512, 2014.

[IS09] Shunji Ito and Taizo Sadahiro. Beta-expansions with negative bases. *Integers*, 9:A22, 239–259, 2009.

[Jac64] Bernard Jacobson. Sums of distinct divisors and sums of distinct units. *Proc. Amer. Math. Soc.*, 15:179–183, 1964.

[JN07] Moshe Jarden and Władysław Narkiewicz. On sums of units. *Monatsh. Math.*, 150(4):327–332, 2007.

[Kal14] Charlene Kalle. Isomorphisms between positive and negative β-transformations. *Ergodic Theory Dynam. Systems*, 34(1):153–170, 2014.

[KS12] Charlene Kalle and Wolfgang Steiner. Beta-expansions, natural extensions and multiple tilings associated with Pisot units. *Trans. Amer. Math. Soc.*, 364(5):2281–2318, 2012.

[MPV11] Zuzana Masáková, Edita Pelantová, and Tomáš Vávra. Arithmetics in number systems with a negative base. *Theoret. Comput. Sci.*, 412(8-10):835–845, 2011.

[MV14] Zuzana Masáková and Tomáš Vávra. Integers in number systems with positive and negative quadratic pisot base. To appear in RAIRO Theor. Inform. Appl., 2014.

[New90] Morris Newman. Consecutive units. *Proc. Amer. Math. Soc.*, 108(2):303–306, 1990.

[Par60] William Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.*, 11:401–416, 1960.

[Rén57] Alfréd Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar*, 8:477–493, 1957.

[Sch80] Klaus Schmidt. On periodic expansions of Pisot numbers and Salem numbers. *Bull. London Math. Soc.*, 12(4):269–278, 1980.

[Śli74] Jan Śliwa. Sums of distinct units. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 22:11–13, 1974.

[Ste12] Wolfgang Steiner. On the structure of $(-\beta)$-integers. *RAIRO Theor. Inform. Appl.*, 46(1):181–200, 2012.

[Ste13] Wolfgang Steiner. Digital expansions with negative real bases. *Acta Math. Hungar.*, 139(1-2):106–119, 2013.

[Thu89] William P. Thurston. *Groups, Tilings and Finite State Automata: Summer 1989 AMS Colloquium Lectures*. Research report GCG. Geometry Computing Group, 1989.

[TZ07] Robert F. Tichy and Volker Ziegler. Units generating the ring of integers of complex cubic fields. *Colloq. Math.*, 109(1):71–83, 2007.

[TZ11] Jörg Thuswaldner and Volker Ziegler. On linear combinations of units with bounded coefficients. *Mathematika*, 57(2):247–262, 2011.

Shrnutí

Tato doktorská práce se věnuje studiu nestandardních reprezentací čísel. Konkrétně se zabýváme dvěma hlavními tématy, pozičním reprezentacím reálných čísel v obecné záporné bázi a dále reprezentacemi algebraických celých čísel v číselných tělesech pomocí algebraických jednotek. Ve speciálním případě využíváme přímou souvislost mezi těmito na první pohled zdánlivě nesouvisejícími reprezentačními problémy.

V první části práce studujeme tzv. $(-\beta)$-rozvoje, definované v práci Ita a Sadahira jako přímá analogie k Rényiho β-rozvojům. Zavádíme zobecnění $(-\beta)$-rozvojů a studujeme jeho vlastnosti, s důrazem na rozhodování o tzv. přípustnosti řetězců cifer. Studujeme strukturu množiny $(-\beta)$-celých čísel, značené $\mathbb{Z}_{-\beta}$, a popisujeme jak množinu délek mezer mezi sousedními prvky v $\mathbb{Z}_{-\beta}$, tak antimorfismy, jejichž pevné body množinu $\mathbb{Z}_{-\beta}$ kódují. Dále ukazujeme výjimečnost třídy tzv. konfluentních Parryho čísel na podobnosti množin β-celých a $(-\beta)$-celých čísel a na jejich blízkém vztahu se zobecněnými spektry.

V druhé části se věnujeme zobecnění tzv. unit sum number problému, neboli rozhodnutí o tom, zda lze všechna algebraická celá čísla v daném číselném tělese vyjádřit jako sumy jednotek. Uvažujeme dvě možná zobecnění, ve kterých sumy jednotek nahrazujeme buď sumami algebraických celých čísel s omezenou normou nebo lineárními kombinacemi jednotek s racionálními koeficienty. U obou zobecnění charakterizujeme případy, ve kterých jsou zmíněné reprezentace celých čísel možné. Nakonec připomeneme definici tzv. DUG-těles, těles v nichž je možné každé algebraické celé číslo reprezentovat sumou vzájemně různých jednotek, a pomocí nových metod rozšíříme seznam totálně komplexních kvartických těles, které jsou DUG-tělesy.